

\mathcal{PT} -symmetric deformations of Calogero models

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ABSTRACT: We demonstrate that Coxeter groups allow for complex \mathcal{PT} -symmetric deformations across the boundaries of all Weyl chambers. We compute the explicit deformations for the A_2 and G_2 -Coxeter group and apply these constructions to Calogero-Moser-Sutherland models invariant under the extended Coxeter groups. The eigenspectra for the deformed models are real and contain the spectra of the undeformed case as subsystem.

1. Introduction

The study of pseudo-Hermitian Hamiltonian systems has attracted a considerable amount of attention in the last few years. For recent reviews and special issues devoted to this topic see [1, 2, 3, 4]. One of the main reasons for the popularity of these types of Hamiltonians is the fact that they possess real eigenvalue spectra, despite of being non-Hermitian, and therefore constitute interesting candidates for a new sort of stable physical systems overlooked up to now. Alternatively to using the concept of pseudo-Hermiticity [5] or quasi-Hermiticity [36, 37] one may equivalently explain the reality of the spectrum of some non-Hermitian Hamiltonians when one encounters unbroken \mathcal{PT} -symmetry, which in the recent context was first pointed out in [6]. Unbroken specifies that both the Hamiltonian *and* the wavefunction remain invariant under a simultaneous parity transformation $\mathcal{P} : x \rightarrow -x$ and time reversal $\mathcal{T} : t \rightarrow -t$. When acting on complex valued functions the anti-linear operator \mathcal{T} is understood to act as complex conjugation.

These observations can be exploited in the construction of new interesting models with real eigenvalue spectra when taking previously studied Hermitian examples as starting points. The above statements imply that one has two possibilities at hand. One could either employ pseudo-Hermiticity, which involves the usually technically difficult task to construct a meaningful metric, e.g. [7, 8, 9, 10, 11, 12], or in contrast use \mathcal{PT} -symmetry as a very transparent and simple principle, at least on the level of the Hamiltonian itself. Starting with a \mathcal{PT} -symmetric Hamiltonian or less restrictive with a parity invariant potential system one may extend such type of models by adding \mathcal{PT} -symmetric terms to it or by

deforming existing terms in a \mathcal{PT} -symmetric manner. The latter construction principle has been applied to a huge number of models, notably the harmonic oscillator in [13], which constitutes the starting point of the current activities in this field of research.

In the context of Calogero models [14, 15, 16] such type of extensions were first carried out in [17, 18] by adding \mathcal{PT} -symmetric terms to the A_n and B_n -Calogero models. Shortly afterwards an alternative procedure was proposed in [19], where the A_2 -Calogero model was genuinely deformed in a \mathcal{PT} -symmetric manner. The analysis in [17] was extended thereafter in [20] to Calogero models related to all Coxeter groups and also generalized to the larger class of Calogero-Moser-Sutherland (CMS) models [14, 15, 16, 21, 22, 23, 24, 25] involving more general types of potentials rather than the rational one. Other versions of deformations of CMS-models have also been proposed for instance in [26], albeit a concrete relation to \mathcal{PT} -symmetry had not been established, even though it is easy to verify that the models constructed in [26] are also \mathcal{PT} -symmetric. The purpose of this paper is to provide the general mathematical framework for the deformation carried out in [19] and generalize the construction to all Coxeter groups and more general potentials. Thereafter we study some of the physical properties of the newly obtained models.

Our manuscript is organized as follows: In order to fix our conventions we recall in section 2 some of the basic features of CMS-models and indicate the structure we expect to find for the deformed models. In section 3 we demonstrate how Coxeter groups may be systematically deformed in a \mathcal{PT} -symmetric manner. We illustrate the general setting with the two explicit examples of the A_2 and G_2 -Coxeter group. We apply these construction in section 4 to CMS-models, which are invariant under the extended Coxeter group. We show that models for which this invariance is broken in a particular way also possess interesting properties. Thereafter we specialize to the Calogero models and construct their eigensystems for some specific deformations. The key finding is that some constraints on the parameter space of the model resulting from physical requirements may be relaxed in the deformed model. For some simple extended model we demonstrate that the energy spectrum is real and contains the one of the undeformed case as a subsystem. We state our conclusions in section 5.

2. Extended symmetries for Calogero-Moser-Sutherland models

Let us briefly recall some features of the CMS-models, which will be relevant for our analysis. The models describe n particles moving on a line, whose coordinates q and canonically conjugate momenta p may be assembled into vectors $q, p \in \mathbb{R}^n$. The Hamiltonian for the CMS-models related to all Coxeter groups \mathcal{W} may be written generically as

$$\mathcal{H}_{\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot q) \quad m, g_\alpha \in \mathbb{R}. \quad (2.1)$$

The dimensionality of the space in which the roots α of the root system Δ are realized is n . The sum in the confining term of the potential only extends over the short roots Δ_s . One may impose further restrictions on the coupling constants g_α in order to guarantee integrability [24, 25, 27] and invariance of the Hamiltonian under the action of \mathcal{W} . The

latter demands that $g_\alpha = g_\beta$ when the roots α and β have the same length, i.e. if $\alpha^2 = \beta^2$. When the potential V is taken to be $V(x) = 1/x^2$ the Hamiltonian (2.1) constitutes the Calogero model, whereas the generalized CMS-models are obtained by choosing $V(x) = 1/\sin^2 x$, $V(x) = 1/\sinh^2 x$ or $V(x) = 1/\operatorname{sn}^2 x$.

A key feature of the model (2.1) for our purposes is that it admits the entire Coxeter group \mathcal{W} as a symmetry, i.e.

$$\begin{aligned} \mathcal{H}_{\text{CMS}} &= \frac{\sigma_i p \cdot \sigma_i p}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \sigma_i q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot \sigma_i q), \\ &= \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\sigma_i^{-1} \alpha \cdot q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\sigma_i^{-1} \alpha \cdot q) \end{aligned} \quad (2.2)$$

where σ_i can be any Weyl reflection (3.1). For the confining term to remain invariant we need to use that short roots are mapped into short root by the entire Coxeter group. This symmetry stipulates that these models are invariant with respect to various parity transformations \mathcal{P} across the hyperplanes through the origin orthogonal to the root α_i or in other words across the boundaries of all Weyl chambers.

Our aim is here to modify the models such that they remain invariant under the action of the newly defined \mathcal{PT} -symmetrically extended Coxeter group, which we denote by $\mathcal{W}^{\mathcal{PT}}$. We propose the new Hamiltonians to be of the form

$$\mathcal{H}_{\mathcal{PT}\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\tilde{\alpha} \in \tilde{\Delta}_s} (\tilde{\alpha} \cdot q)^2 + \frac{1}{2} \sum_{\tilde{\alpha} \in \tilde{\Delta}} g_{\tilde{\alpha}} V(\tilde{\alpha} \cdot q) \quad m, g_{\tilde{\alpha}} \in \mathbb{R}, \quad (2.3)$$

where we have replaced the standard roots $\alpha \in \Delta$ by deformed roots $\tilde{\alpha} \in \tilde{\Delta}$. Formally \mathcal{H}_{CMS} and $\mathcal{H}_{\mathcal{PT}\text{CMS}}$ are very similar, with the crucial difference that the latter is in general complex and non-Hermitian.

Nonetheless, the \mathcal{PT} -symmetry can be utilised to establish the reality of the spectrum with a minor modification. As we have complexified here each Weyl reflection across any hyperplane orthogonal to every root we have as many \mathcal{PT} -operators, i.e. anti-linear operators, as hyperplanes. This means we can employ any of these operators in the standard argument¹. In turn this also means that we could in principle make our construction less constraining by demanding less symmetry. What is of course not known at this point is whether the wavefunctions of the deformed Hamiltonian also respect the extended symmetry. However, as we shall demonstrate below with some concrete examples this will indeed be the case.

In order to see that such type of models really exist and how these models can be constructed we need to assemble first some mathematical tools and establish the fact that one can indeed construct a meaningful set of deformed roots $\tilde{\alpha}$.

¹By construction $\tilde{\sigma}_{\alpha_i}$, see (3.5) for definition, is a symmetry of the new Hamiltonian $\mathcal{H}_{\mathcal{PT}\text{CMS}}$, that is we have $[\mathcal{H}_{\mathcal{PT}\text{CMS}}, \tilde{\sigma}_{\alpha_i}] = 0$. Assuming further that the eigenfunctions are also invariant with regard to $\mathcal{W}^{\mathcal{PT}}$, i.e. $\tilde{\sigma}_{\alpha_i} \Phi = \Phi$, the reality of the eigenspectrum follows trivially from

$$\varepsilon \Phi = \mathcal{H}_{\mathcal{PT}\text{CMS}} \Phi = \mathcal{H}_{\mathcal{PT}\text{CMS}} \tilde{\sigma}_{\alpha_i} \Phi = \tilde{\sigma}_{\alpha_i} \mathcal{H}_{\mathcal{PT}\text{CMS}} \Phi = \tilde{\sigma}_{\alpha_i} \varepsilon \Phi = \varepsilon^* \tilde{\sigma}_{\alpha_i} \Phi = \varepsilon^* \Phi.$$

3. \mathcal{PT} -symmetric deformations of Coxeter groups

We recall, see e.g. [28, 29], that a Coxeter group \mathcal{W} is generated by the Weyl reflections σ_i associated with a set of simple roots $\{\alpha_i\}$ which span the entire root space Δ

$$\sigma_i(x) = x - 2 \frac{x \cdot \alpha_i}{\alpha_i^2} \alpha_i \quad \text{for } 1 \leq i \leq \ell \equiv \text{rank } \mathcal{W}; \ x, \alpha_i \in \mathbb{R}^n. \quad (3.1)$$

The roots may be represented in various different Euclidean spaces with dimensionality not necessarily equal to ℓ . Here our aim is to construct a complex extended root system $\tilde{\Delta}(\varepsilon)$ containing the roots $\tilde{\alpha}_i(\varepsilon)$ represented in $\mathbb{R}^n \oplus i\mathbb{R}^n$, depending on some deformation parameter $\varepsilon \in \mathbb{R}$. We demand that each deformed root reduces one-to-one to a root in the root space Δ

$$\lim_{\varepsilon \rightarrow 0} \tilde{\alpha}_i(\varepsilon) = \alpha_i \quad \text{for } \tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon), \alpha_i \in \Delta, \quad (3.2)$$

such that the entire root space reduces as

$$\lim_{\varepsilon \rightarrow 0} \tilde{\Delta}(\varepsilon) = \Delta. \quad (3.3)$$

Furthermore, we require that the extended root system $\tilde{\Delta}(\varepsilon)$ remains invariant under the \mathcal{PT} -symmetrically extended Coxeter group $\mathcal{W}^{\mathcal{PT}}$. Note that in principle we may choose any of the hyperplanes through the origin orthogonal to a root $\alpha_i \in \Delta$ across which the parity symmetry \mathcal{P} can be extended to a \mathcal{PT} -symmetry. Thus we could expect $\ell h \cdot \ell h/2$ deformed roots, with h denoting the Coxeter number and ℓh being the total number of roots. However, the deformations to any of the hyperplanes can in fact be made equivalent and the replacement

$$\alpha_i \rightarrow \tilde{\alpha}_i(\varepsilon) \quad \text{for } 1 \leq i \leq \ell h, \quad (3.4)$$

becomes indeed one-to-one as we shall see below.

From these requirements we may now attempt to construct the root system $\tilde{\Delta}(\varepsilon)$. We start by selecting a particular root α_i , which does not have to be simple, and perform a complex \mathcal{PT} -symmetric extension across the hyperplane through the origin orthogonal to this root. This deformation leads to a new, so far unspecified root $\tilde{\alpha}_i(\varepsilon)$. Studying now the properties of this root will enable us to determine it. Decomposing the complex extended Weyl reflection into a product of standard Weyl reflections (3.1) and a complex conjugation (time-reversal) as

$$\tilde{\sigma}_{\alpha_i} := \sigma_{\alpha_i} \mathcal{T}, \quad (3.5)$$

we compute its action on a root

$$\tilde{\sigma}_{\alpha_j}(\tilde{\alpha}_j(\varepsilon)) = \sigma_{\alpha_j} \mathcal{T}(\text{Re } \tilde{\alpha}_j(\varepsilon)) + \sigma_{\alpha_j} \mathcal{T}(i \text{Im } \tilde{\alpha}_j(\varepsilon)) \quad (3.6)$$

$$= \sigma_{\alpha_j}(\text{Re } \tilde{\alpha}_j(\varepsilon)) - i \sigma_{\alpha_j}(\text{Im } \tilde{\alpha}_j(\varepsilon)) \quad (3.7)$$

$$= -\text{Re } \tilde{\alpha}_j(\varepsilon) - i \text{Im } \tilde{\alpha}_j(\varepsilon) \quad (3.8)$$

$$= -\tilde{\alpha}_j(\varepsilon). \quad (3.9)$$

In view of (3.2) we demanded here that the complex extended Weyl reflection $\tilde{\sigma}_{\alpha_i}$ maps the deformed root $\tilde{\alpha}_i(\varepsilon)$ into its negative which should in view of the limit (3.2) also hold for

the real part independently. For the remaining term of the root the minus sign is created by the complex conjugation \mathcal{T} , such the imaginary part has to be invariant under the Weyl reflection, i.e. it has to be a vector lying in the hyperplane across which the reflection is carried out. Comparing now (3.8) and (3.9) we find as solution for $\tilde{\alpha}_i(\varepsilon)$

$$\operatorname{Re} \tilde{\alpha}_i(\varepsilon) = R(\varepsilon)\alpha_i \quad \text{and} \quad \operatorname{Im} \tilde{\alpha}_i(\varepsilon) = I(\varepsilon) \sum_{j \neq i} \kappa_j \lambda_j, \quad (3.10)$$

where $\kappa_j \in \mathbb{R}$ and the λ_i have to be elements of the weight lattice, i.e. they are orthogonal to the simple roots $2\lambda_i \cdot \alpha_j / \alpha_j^2 = \delta_{ij}$. The real valued functions $R(\varepsilon)$ and $I(\varepsilon)$ are arbitrary at this stage, with the only condition to satisfy

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} I(\varepsilon) = 0, \quad (3.11)$$

in order to fulfill the requirement (3.2). Note that $R(\varepsilon)$ and $I(\varepsilon)$ may also be multiplied by any invariant of the extended Weyl group $\mathcal{W}^{\mathcal{PT}}$.

The remaining roots can be constructed by acting with all possible non-equivalent $\ell h - 1$ reflections $\sigma_{\alpha_i} \mathcal{T}$ on these roots and hence producing the anticipated number of ℓh roots $\tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon)$. Supposing now we have constructed a new root as $\beta = \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_i)$, it is then clear that by construction also for that new root the imaginary part is orthogonal to its real part

$$\operatorname{Re} \beta \cdot \operatorname{Im} \beta = \sigma_{\alpha_k}(\operatorname{Re} \tilde{\alpha}_i) \cdot \sigma_{\alpha_k}(\operatorname{Im} \tilde{\alpha}_i) = \operatorname{Re} \tilde{\alpha}_i \cdot \operatorname{Im} \tilde{\alpha}_i = 0. \quad (3.12)$$

The property which is, however, not guaranteed is that the decomposition of the undeformed root into a sum over simple roots $\alpha = \sum_{i=1} \alpha_i$ is preserved by the deformation. Nonetheless, we shall verify this feature for the explicit examples below. Sometimes we can even find

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_i) \cdot \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_j), \quad (3.13)$$

for which there is also no general justification. When (3.13) holds we can even impose a stronger constraint and require that inner products of roots and deformed roots are identical

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j, \quad (3.14)$$

which allows us to fix the functions $R(\varepsilon)$ and $I(\varepsilon)$.

Alternatively to the above construction we may also deform each root as

$$\alpha_i \rightarrow \tilde{\alpha}_i(\varepsilon) = R(\varepsilon)\alpha_i \pm iI(\varepsilon)\alpha_i \quad \text{for } \alpha_i \in \Delta_{\pm}. \quad (3.15)$$

Note that in this deformation positive and negative roots in $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ are no longer related by an overall minus sign as in Δ_+ and Δ_- , where $\alpha_i \in \Delta_+$ always has a counter part $-\alpha_i \in \Delta_-$. However, by construction we still have the property

$$\tilde{\sigma}_{\alpha_i}(\tilde{\alpha}_i(\varepsilon)) = -\tilde{\alpha}_i(\varepsilon), \quad (3.16)$$

which is needed to achieve invariance under the extended Coxeter group $\mathcal{W}^{\mathcal{PT}}$. Now, unlike as in the previous construction, the minus sign for the imaginary part is created by

definition and not by the action of $\tilde{\sigma}_{\alpha_i}$. In general, we may encounter the four possibilities

$$\tilde{\sigma}_{\alpha_i}(\tilde{\alpha}_j(\varepsilon)) \in \pm\tilde{\Delta}_- \quad \text{for } \sigma_{\alpha_i}(\alpha_j(\varepsilon)) \in \Delta_{\mp}, \tilde{\alpha}_j(\varepsilon) \in \tilde{\Delta}_+, \quad (3.17)$$

$$\tilde{\sigma}_{\alpha_i}(\tilde{\alpha}_j(\varepsilon)) \in \pm\tilde{\Delta}_+ \quad \text{for } \sigma_{\alpha_i}(\alpha_j(\varepsilon)) \in \Delta_{\pm}, \tilde{\alpha}_j(\varepsilon) \in \tilde{\Delta}_-. \quad (3.18)$$

Thus any root $\tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}$ of the form (3.15) is guaranteed to be mapped into $\pm\tilde{\Delta}$ under the action of $\mathcal{W}^{\mathcal{PT}}$, which means the deformed root system remains only invariant up to an overall sign. However, in our application below overall signs are irrelevant so that the deformation (3.15) will be suitable for the application in mind.

Let us now verify that the procedure outlined above indeed leads to a closed \mathcal{PT} -symmetrically extended Weyl group $\mathcal{W}^{\mathcal{PT}}$ for some concrete examples.

3.1 \mathcal{PT} -symmetric deformations of the A_2 -Coxeter group

We recall first the action of the Weyl reflections on the simple roots by computing (3.1) with the Cartan matrix $K_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$, whose entries are $K_{11} = K_{22} = 2$, $K_{12} = K_{21} = -1$. The combinations of Weyl reflections achieving a reflection across the hyperplanes through the origin orthogonal to the three positive roots α_1 , α_2 and $\alpha_1 + \alpha_2$ of A_2 are

$$\begin{aligned} \sigma_1 : \alpha_1 &\mapsto -\alpha_1, & \alpha_2 &\mapsto \alpha_1 + \alpha_2, & \alpha_1 + \alpha_2 &\mapsto \alpha_2, \\ \sigma_2 : \alpha_1 &\mapsto \alpha_1 + \alpha_2, & \alpha_2 &\mapsto -\alpha_2, & \alpha_1 + \alpha_2 &\mapsto \alpha_1, \\ \sigma_1\sigma_2\sigma_1 : \alpha_1 &\mapsto -\alpha_2, & \alpha_2 &\mapsto -\alpha_1, & \alpha_1 + \alpha_2 &\mapsto -\alpha_1 - \alpha_2. \end{aligned} \quad (3.19)$$

As a starting point for the deformation we choose the simple root α_1 and extend the parity symmetry across the hyperplane through the origin orthogonal to this root. According to (3.10) the deformed root should be taken to

$$\tilde{\alpha}_1(\varepsilon) := R(\varepsilon)\alpha_1 \pm iI(\varepsilon)\lambda_2, \quad (3.20)$$

where we introduced the fundamental weight $\lambda_2 = (\alpha_1 + 2\alpha_2)/3$. Next we compute the action of the complex reflections $\tilde{\sigma}_{\alpha_i}$ on this root in order to construct the remaining five deformed roots. By construction we have

$$\tilde{\sigma}_1\tilde{\alpha}_1(\varepsilon) = -R(\varepsilon)\alpha_1 \mp iI(\varepsilon)\lambda_2 =: -\tilde{\alpha}_1(\varepsilon). \quad (3.21)$$

Having determined $\pm\tilde{\alpha}_1$ we may calculate the deformations of α_2 from $\mp\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\alpha}_1$ guided by the undeformed case (3.19). We compute

$$-\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_2 \mp iI(\varepsilon)\lambda_1 =: \tilde{\alpha}_2(\varepsilon) \quad (3.22)$$

where we obtained the fundamental weight $\lambda_1 = (2\alpha_1 + \alpha_2)/3$. We may verify that the remaining reflections in (3.19) indeed yield a consistent system

$$\tilde{\sigma}_2\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)(\alpha_1 + \alpha_2) \mp iI(\varepsilon)(\lambda_1 - \lambda_2) = \tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon), \quad (3.23)$$

$$\tilde{\sigma}_1\tilde{\alpha}_2(\varepsilon) = R(\varepsilon)(\alpha_1 + \alpha_2) \mp iI(\varepsilon)(\lambda_1 - \lambda_2) = \tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon), \quad (3.24)$$

$$\tilde{\sigma}_2\tilde{\alpha}_2(\varepsilon) = -R(\varepsilon)\alpha_2 \pm iI(\varepsilon)\lambda_1 = -\tilde{\alpha}_2(\varepsilon), \quad (3.25)$$

$$\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\alpha}_2(\varepsilon) = -R(\varepsilon)\alpha_1 \mp iI(\varepsilon)\lambda_2 = -\tilde{\alpha}_1(\varepsilon). \quad (3.26)$$

We may verify that in (3.23) and (3.24) the imaginary part of the deformed root $\tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon)$ is indeed orthogonal to the root $\alpha_1 + \alpha_2$ as it should be by construction. Alternatively we could have started with the expressions (3.20) and (3.22) involving the ambiguities of the relative signs in front of the imaginary parts and the unknown functions $R(\varepsilon)$ and $I(\varepsilon)$. The subsequent action of combinations of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ would fix the sign ambiguity and produce the same set of deformed roots. Note that we also have the property

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_i) \cdot \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_j), \quad i, j, k = 1, 2. \quad (3.27)$$

If we impose the additional constraint that inner products of root and deformed roots are identical

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j, \quad (3.28)$$

we may fix the deformation functions to $R(\varepsilon) = \cosh \varepsilon$, $I(\varepsilon) = \sqrt{3} \sinh \varepsilon$. The factor of $\sqrt{3}$ in the function $I(\varepsilon)$ is somewhat natural as it ensures that the roots in the real part of the deformed roots $\tilde{\alpha}_i(\varepsilon)$ and the weights in the imaginary part have the same length. As intended, we have achieved a simple one-to-one relation between (3.19) and the corresponding identities for the deformed system simply by replacing $\sigma_i \rightarrow \tilde{\sigma}_i$ and $\alpha_i \rightarrow \tilde{\alpha}_i(\varepsilon)$. We depict the roots and the hyperplanes in figure 1.

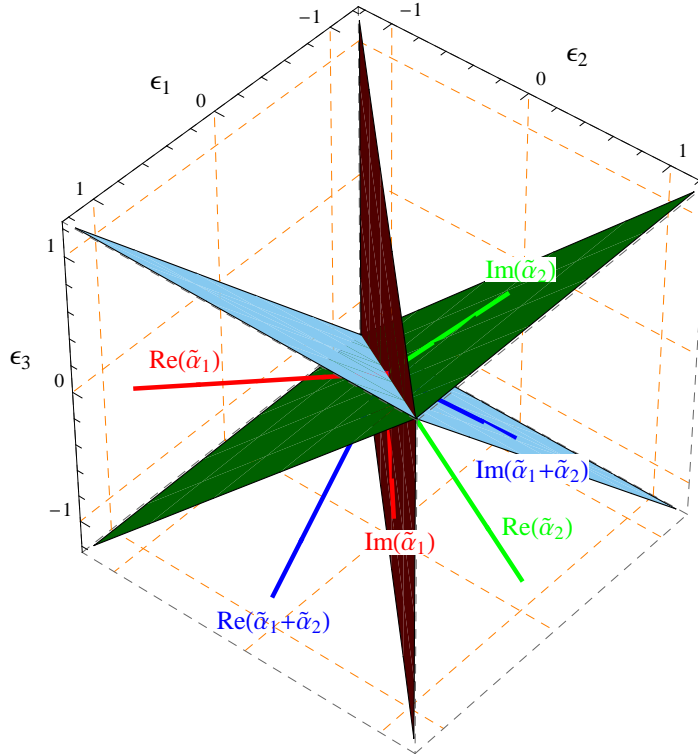


Figure 1: Real and imaginary parts of the A_2 deformed roots divided by $R(\varepsilon)$ and $I(\varepsilon)$, respectively, in the three dimensional standard representation for the simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$. Both parts of a particular positive root $\tilde{\alpha}_i$ are depicted in the same colour (on-line).

Alternatively we can deform the six roots according to the principle (3.16) as

$$\pm\alpha_1 \rightarrow \tilde{\alpha}_1^\pm = \pm R(\varepsilon)\alpha_1 + iI(\varepsilon)\alpha_1 \quad (3.29)$$

$$\pm\alpha_2 \rightarrow \tilde{\alpha}_2^\pm = \pm R(\varepsilon)\alpha_2 + iI(\varepsilon)\alpha_2 \quad (3.30)$$

$$\pm(\alpha_1 + \alpha_2) \rightarrow \tilde{\alpha}_1^\pm + \tilde{\alpha}_2^\pm = \pm R(\varepsilon)(\alpha_1 + \alpha_2) + iI(\varepsilon)(\alpha_1 + \alpha_2). \quad (3.31)$$

As pointed out we no longer have $\tilde{\alpha}_1^- = -\tilde{\alpha}_1^+$. Nonetheless, it is easy to verify that these roots are mapped into each other by \mathcal{W}^{PT} as

$$\begin{aligned} \tilde{\sigma}_1 : \tilde{\alpha}_1^+ &\mapsto \tilde{\alpha}_1^-, & \tilde{\alpha}_2^+ &\mapsto -(\tilde{\alpha}_1^- + \tilde{\alpha}_2^-), & \tilde{\alpha}_1^+ + \tilde{\alpha}_2^+ &\mapsto -\tilde{\alpha}_2^-, \\ \tilde{\sigma}_2 : \tilde{\alpha}_1^+ &\mapsto -(\tilde{\alpha}_1^- + \tilde{\alpha}_2^-), & \tilde{\alpha}_2^+ &\mapsto -\tilde{\alpha}_2^-, & \tilde{\alpha}_1^+ + \tilde{\alpha}_2^+ &\mapsto -\tilde{\alpha}_1^-, \\ \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1 : \tilde{\alpha}_1^+ &\mapsto \tilde{\alpha}_2^-, & \tilde{\alpha}_2^+ &\mapsto \tilde{\alpha}_1^-, & \tilde{\alpha}_1^+ + \tilde{\alpha}_2^+ &\mapsto \tilde{\alpha}_1^- + \tilde{\alpha}_2^-. \end{aligned} \quad (3.32)$$

For these roots the inner product is not preserved and (3.27) does not hold in this case.

3.2 PT -symmetric deformations of the G_2 -Coxeter group

Since only roots of one length emerge in root systems related to simply laced Lie algebras, some features discussed this far are slightly different for non-simply laced cases. Let us therefore present one explicitly example in order to exhibit the differences. We recall, see e.g. [28, 29], that the set of roots invariant under the G_2 -Coxeter group separates into a set of short and long roots Δ_s and Δ_l , respectively,

$$\Delta = \Delta_s \cup \Delta_l = \pm\{\alpha_1, (\alpha_1 + \alpha_2), (2\alpha_1 + \alpha_2)\} \cup \pm\{\alpha_2, (3\alpha_1 + \alpha_2), (3\alpha_1 + 2\alpha_2)\}. \quad (3.33)$$

Using the Cartan matrix with entries $K_{11} = K_{22} = 2$, $K_{12} = -1$ and $K_{21} = -3$ we may compute the action of the Weyl reflections on the simple roots by evaluating (3.1). The combinations of Weyl reflections achieving a reflection across the hyperplanes through the origin orthogonal to the six positive roots are presented in the following table:

	α_1	$\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$	α_2	$3\alpha_1 + \alpha_2$	$3\alpha_1 + 2\alpha_2$
$\sigma_1 :$	$-\alpha_1$	$2\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$	$3\alpha_1 + \alpha_2$	α_2	$3\alpha_1 + 2\alpha_2$
$\sigma_2 :$	$\alpha_1 + \alpha_2$	α_1	$2\alpha_1 + \alpha_2$	$-\alpha_2$	$3\alpha_1 + 2\alpha_2$	$3\alpha_1 + \alpha_2$
$\sigma_2\sigma_1\sigma_2 :$	$2\alpha_1 + \alpha_2$	$-\alpha_1 - \alpha_2$	α_1	$-3\alpha_1 - 2\alpha_2$	$3\alpha_1 + \alpha_2$	$-\alpha_2$
$\sigma_1\sigma_2\sigma_1 :$	$-2\alpha_1 - \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_1$	$3\alpha_1 + 2\alpha_2$	$-3\alpha_1 - \alpha_2$	α_2
$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 :$	$-\alpha_1 - \alpha_2$	$-\alpha_1$	$-2\alpha_1 - \alpha_2$	α_2	$-3\alpha_1 - 2\alpha_2$	$-3\alpha_1 - \alpha_2$
$\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 :$	α_1	$-2\alpha_1 - \alpha_2$	$-\alpha_1 - \alpha_2$	$-3\alpha_1 - \alpha_2$	$-\alpha_2$	$-3\alpha_1 - 2\alpha_2$

Table 1: Simple Weyl reflections acting on the six positive roots of G_2

Having assembled the key properties for the undeformed root system, we choose as a starting point for the construction of $\tilde{\Delta}$ the deformation of the simple roots α_1 or α_2 and extend the parity symmetry across the hyperplane through the origin orthogonal to these roots. According to (3.10) the deformed counterparts can be taken to be

$$\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_1 \pm iI(\varepsilon)\lambda_2, \quad (3.34)$$

$$\tilde{\alpha}_2(\varepsilon) = R(\varepsilon)\alpha_2 \mp i3I(\varepsilon)\lambda_1, \quad (3.35)$$

where we used the two fundamental weights $\lambda_1 = 2\alpha_1 + \alpha_2$ and $\lambda_2 = 3\alpha_1 + 2\alpha_2$ of G_2 . Acting now with products of the complex reflections $\tilde{\sigma}_{\alpha_i}$ first on $\tilde{\alpha}_1(\varepsilon)$ yields the deformations of the short roots

$$\tilde{\sigma}_1 \tilde{\alpha}_1(\varepsilon) = -R(\varepsilon)\alpha_1 \mp iI(\varepsilon)\lambda_2 = -\tilde{\alpha}_1(\varepsilon), \quad (3.36)$$

$$\tilde{\sigma}_2 \tilde{\alpha}_1(\varepsilon) = R(\varepsilon)(\alpha_1 + \alpha_2) \mp iI(\varepsilon)(3\lambda_1 - \lambda_2) = \tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon), \quad (3.37)$$

$$\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\alpha}_1(\varepsilon) = -R(\varepsilon)(2\alpha_1 + \alpha_2) \mp iI(\varepsilon)(3\lambda_1 - 2\lambda_2) = -2\tilde{\alpha}_1(\varepsilon) - \tilde{\alpha}_2(\varepsilon), \quad (3.38)$$

$$\tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\alpha}_1(\varepsilon) = R(\varepsilon)(2\alpha_1 + \alpha_2) \mp iI(\varepsilon)(3\lambda_1 - 2\lambda_2) = 2\tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon), \quad (3.39)$$

$$\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\alpha}_1(\varepsilon) = -R(\varepsilon)(\alpha_1 + \alpha_2) \pm iI(\varepsilon)(3\lambda_1 - \lambda_2) = -\tilde{\alpha}_1(\varepsilon) - \tilde{\alpha}_2(\varepsilon), \quad (3.40)$$

$$\tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_1 \pm iI(\varepsilon)\lambda_2 = \tilde{\alpha}_1(\varepsilon). \quad (3.41)$$

The action of products of reflections $\tilde{\sigma}_{\alpha_i}$ on $\tilde{\alpha}_2(\varepsilon)$ yields the deformations of the long roots

$$\tilde{\sigma}_1 \tilde{\alpha}_2(\varepsilon) = R(\varepsilon)(3\alpha_1 + \alpha_2) \mp i3I(\varepsilon)(\lambda_1 - \lambda_2) = 3\tilde{\alpha}_1(\varepsilon) + \tilde{\alpha}_2(\varepsilon), \quad (3.42)$$

$$\tilde{\sigma}_2 \tilde{\alpha}_2(\varepsilon) = -R(\varepsilon)\alpha_2 \pm i3I(\varepsilon)\lambda_1 = -\tilde{\alpha}_2(\varepsilon), \quad (3.43)$$

$$\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\alpha}_2(\varepsilon) = R(\varepsilon)(3\alpha_1 + 2\alpha_2) \mp i3I(\varepsilon)(2\lambda_1 - \lambda_2) = 3\tilde{\alpha}_1(\varepsilon) + 2\tilde{\alpha}_2(\varepsilon), \quad (3.44)$$

$$\tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\alpha}_2(\varepsilon) = -R(\varepsilon)(3\alpha_1 + 2\alpha_2) \pm i3I(\varepsilon)(2\lambda_1 - \lambda_2) = -3\tilde{\alpha}_1(\varepsilon) - 2\tilde{\alpha}_2(\varepsilon), \quad (3.45)$$

$$\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\alpha}_2(\varepsilon) = R(\varepsilon)\alpha_2 \mp i3I(\varepsilon)\lambda_1 = \tilde{\alpha}_2(\varepsilon), \quad (3.46)$$

$$\tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\alpha}_2(\varepsilon) = -R(\varepsilon)(3\alpha_1 + \alpha_2) \pm i3I(\varepsilon)(\lambda_1 - \lambda_2) = -3\tilde{\alpha}_1(\varepsilon) - \tilde{\alpha}_2(\varepsilon). \quad (3.47)$$

For a particular representation we depict the constructed roots in figure 2.

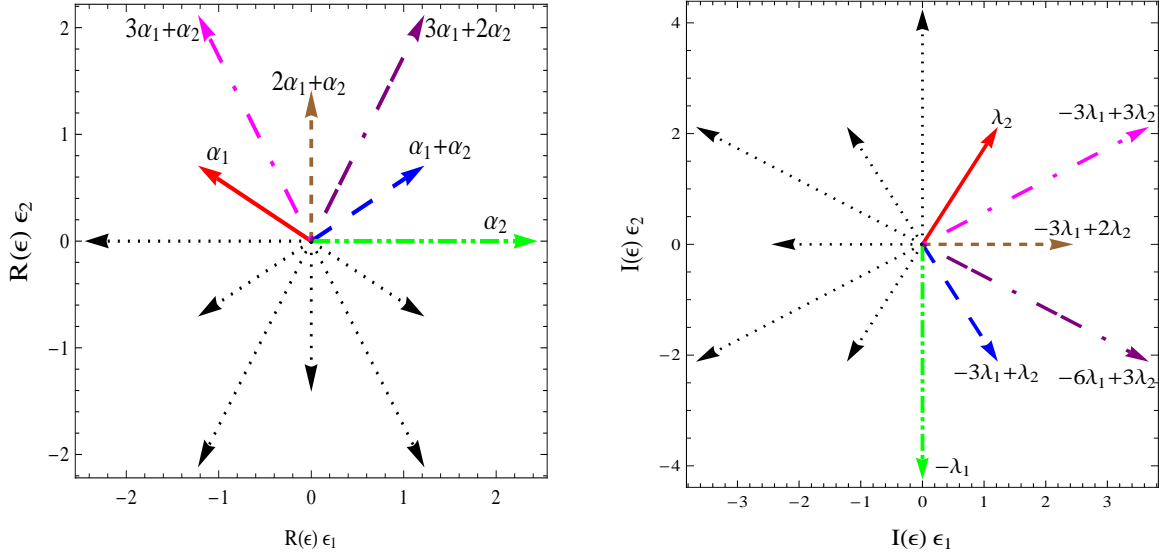


Figure 2: Real and imaginary parts of the G_2 -deformed roots in the two dimensional basis for the simple roots $\alpha_1 = (\varepsilon_2 - \sqrt{3}\varepsilon_1)/\sqrt{2}$, $\alpha_2 = \sqrt{6}\varepsilon_1$. Both parts of a particular positive root $\tilde{\alpha}_i$ are depicted in the same line style (on-line also colour).

As it should be by construction, we can check for consistence once more that indeed the imaginary part is orthogonal to the real part of each deformed root. Again we observe

the property

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_i) \cdot \tilde{\sigma}_{\alpha_k}(\tilde{\alpha}_j), \quad i, j, k = 1, 2 \quad (3.48)$$

and with the additional requirement

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j, \quad (3.49)$$

we may fix the deformation functions to $R(\varepsilon) = \cosh \varepsilon$, $I(\varepsilon) = 1/\sqrt{3} \sinh \varepsilon$. We have achieved a simple one-to-one relation between (3.19) and the corresponding identities for the deformed system simply by replacing $\sigma_i \rightarrow \tilde{\sigma}_i$ and $\alpha_i \rightarrow \tilde{\alpha}_i(\varepsilon)$.

Clearly we can also choose the deformation according to (3.15) as for the A_2 -case, but we will not report this here.

4. \mathcal{PT} -symmetric deformations of Calogero-Moser-Sutherland models

Taking the previous remarks into account it is now straightforward to formulate new types of CMS-models, which are invariant under the action of \mathcal{PT} -symmetrically extended Weyl groups $\mathcal{W}^{\mathcal{PT}}$. The Hamiltonian will be of the form $\mathcal{H}_{\mathcal{PT}\text{CMS}}$ as specified in (2.3). Let us study some concrete examples.

4.1 \mathcal{PT} -symmetrically deformed A_2 -Calogero-Moser-Sutherland models

Beyond the two particle problem the A_2 -CMS model is the simplest classical example, constituting in some representation the three-body problem with a two particle interaction [16]. For this Coxeter group we consider now the Hamiltonian $\mathcal{H}_{\mathcal{PT}\text{CMS}}$ in (2.3) with the two simple roots taken in the standard three dimensional representation $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$, with ε_i being an orthogonal basis in \mathbb{R}^3 with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$ and the dynamical variables to be $q = \{q_1, q_2, q_3\}$. Using then the deformed roots as constructed in (3.20)-(3.26), the potential of the \mathcal{PT} -symmetrically extended model acquires the form

$$V_{\mathcal{PT}\text{CMS}}^{A_2} = g \sum_{\substack{1 \leq j < k \leq 3 \\ j, k \neq l}} V[R(\varepsilon)(q_j - q_k) + i(-1)^{j+k}I(\varepsilon)(q_j + q_k - 2q_l)], \quad (4.1)$$

where $V(x)$ can be of Calogero type, i.e. $V(x) = 1/x^2$ or any of the functions $1/\sin^2 x$, $1/\sinh^2 x$, $1/\text{sn}^2 x$.

By construction these potentials are symmetric with regard to $\mathcal{W}^{\mathcal{PT}}$, which of course can also be seen explicitly for the dynamical variables $\tilde{\sigma}_{\alpha_1} \equiv q_1 \leftrightarrow q_2, i \rightarrow -i$, $\tilde{\sigma}_{\alpha_2} \equiv q_2 \leftrightarrow q_3, i \rightarrow -i$ and $\tilde{\sigma}_{\alpha_1+\alpha_2} \equiv q_1 \leftrightarrow q_3, i \rightarrow -i$.

Instead of the three dimensional representation we may also represent the roots in a two dimensional space, i.e. $\alpha_1 = \sqrt{2}\varepsilon_1$, $\alpha_2 = \sqrt{3/2}\varepsilon_2 - \sqrt{2}\varepsilon_1$ and express the dynamical variables in terms of Jacobi relative coordinates $q = \{X, Y\}$. Comparison between the two representations then leads to the well known relations between the different sets of variables $X = (q_1 - q_2)/\sqrt{2}$ and $Y = (q_1 + q_2 - 2q_3)/\sqrt{6}$. The third coordinate is usually taken to be the center-of-mass coordinate $R = (q_1 + q_2 + q_3)/3$. Moreover, it is convenient to parameterize X and Y further in terms of polar coordinates $X = r \sin \phi$, $Y = r \cos \phi$. In

this formulation the relations for the potential simplify even more with the special choice $R(\varepsilon) = \cosh \varepsilon$ and $I(\varepsilon) = \sqrt{3} \sinh \varepsilon$ as already mentioned after (3.28). With these choices the potential (4.1) is transformed into

$$V_{\mathcal{PT}\text{CMS}}^{A_2} = g \sum_{k=-1,0,1} V \left[\sqrt{2}r \sin(\phi - i\varepsilon + k\frac{2\pi}{3}) \right]. \quad (4.2)$$

Taking the special case $V(x) = g/x^2$ the version (4.2) of the \mathcal{PT} -symmetrically extended A_2 -Calogero model is essentially the potential suggested in [19], where it was obtained by deforming directly the Calogero model in the form (4.2) for $\varepsilon = 0$ across the symmetry $\phi \rightarrow -\phi$ via the recipe $\phi \mapsto \phi - i\varepsilon$. We have demonstrated here how to obtain it as a special case from a more general and systematic setting. The virtue of the version (4.2) in the new coordinate system is that it leads to a separable Schrödinger equation. In section 4.3 we make use of this fact and investigate some properties of the model, notably to construct its eigenfunctions and eigenvalues.

Clearly we may also choose the deformations according to the alternative deformation (3.15), in which case the \mathcal{PT} -symmetrically extended model is of the form

$$V_{\mathcal{PT}\text{CMS}}^{A_2} = \frac{g}{2} \sum_{1 \leq j < k \leq 3} V[(R(\varepsilon) + iI(\varepsilon))(q_j - q_k)] + \frac{g}{2} \sum_{1 \leq j < k \leq 3} V[(R(\varepsilon) - iI(\varepsilon))(q_j - q_k)], \quad (4.3)$$

when choosing the roots to be in the standard representation. Note that, whereas in the undeformed case the contributions from any negative roots equals the one resulting from its positive counterpart, now these roots give different contributions. Expressing (4.3) in terms of Jacobian relative coordinates and making in addition the choice $R(\varepsilon) = 1$ and $I(\varepsilon) = \varepsilon/r$ the potential simply becomes

$$V_{\mathcal{PT}\text{CMS}}^{A_2} = \frac{g}{2} \sum_{k=0,\pm 1} \left[V \left[\sqrt{2}(r + i\varepsilon) \sin(\phi + \frac{2\pi}{3}k) \right] + V \left[\sqrt{2}(r - i\varepsilon) \sin(\phi + \frac{2\pi}{3}k) \right] \right]. \quad (4.4)$$

Note that in the choice for $I(\varepsilon)$ we made use fact that we can multiply this quantity by any invariant of $\mathcal{W}^{\mathcal{PT}}$. Clearly $r = \sqrt{(\alpha_1 \cdot q)^2/3 + (\alpha_2 \cdot q)^2/3 + (\alpha_1 \cdot q + \alpha_2 \cdot q)^2/3}$ is such an invariant. Thus when we restrict the sum in (4.3), (4.4) to the positive or negative roots only the deformation is simply achieved by $r \mapsto r + i\varepsilon$ or $r \mapsto r - i\varepsilon$, respectively. This corresponds to the deformation of the symmetry $r \rightarrow -r$. One should note that the restriction to just half of the number of roots will break the invariance under the action of $\mathcal{W}^{\mathcal{PT}}$.

4.2 \mathcal{PT} -symmetrically deformed G_2 -Calogero-Moser-Sutherland models

The G_2 -CMS-model, constitutes a further standard example, since it can be viewed as the classical three-body problem with a two and a three-body interaction term [30]. As in the previous subsection we may now realize the roots in various different ways. Either we can take the so-called standard three dimensional representation for the simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ as concrete realization for the simple roots of G_2 in \mathbb{R}^3 and the dynamical variables to be $q = \{q_1, q_2, q_3\}$ or alternatively we may also represent

them in a two dimensional space as $\alpha_1 = (\varepsilon_2 - \sqrt{3}\varepsilon_1)/\sqrt{2}$, $\alpha_2 = \sqrt{6}\varepsilon_1$ and express the dynamical variables in terms of Jacobi relative coordinates $q = \{X, Y\}$. Once again the comparison between the two representations yields to the same relations for the Jacobi relative coordinates $X = (q_1 - q_2)/\sqrt{2}$ and $Y = (q_1 + q_2 - 2q_3)/\sqrt{6}$. Explicitly the inner products in all coordinate systems are computed to

$$\alpha_1 \cdot q = q_1 - q_2 = \sqrt{2}X = \sqrt{2}r \sin \phi, \quad (4.5)$$

$$(\alpha_1 + \alpha_2) \cdot q = q_3 - q_1 = -\frac{1}{\sqrt{2}}(\sqrt{3}Y + X) = -\sqrt{2}r \sin(\frac{2\pi}{3} - \phi), \quad (4.6)$$

$$(2\alpha_1 + \alpha_2) \cdot q = q_3 - q_2 = -\frac{1}{\sqrt{2}}(\sqrt{3}Y - X) = -\sqrt{2}r \sin(\frac{2\pi}{3} + \phi), \quad (4.7)$$

$$\alpha_2 \cdot q = q_2 + q_3 - 2q_1 = -\sqrt{\frac{3}{2}}(\sqrt{3}X + Y) = \sqrt{6}r \cos(\frac{2\pi}{3} + \phi), \quad (4.8)$$

$$(3\alpha_1 + \alpha_2) \cdot q = q_1 + q_3 - 2q_2 = \sqrt{\frac{3}{2}}(\sqrt{3}X - Y) = \sqrt{6}r \cos(\frac{2\pi}{3} - \phi), \quad (4.9)$$

$$(3\alpha_1 + 2\alpha_2) \cdot q = 2q_3 - q_1 - q_2 = -\sqrt{6}Y = -\sqrt{6}r \cos \phi. \quad (4.10)$$

The expressions for the short roots (4.5), (4.6) and (4.7) just yield the expressions for the A_2 -roots α_1 , $-\alpha_2$ and $-\alpha_1 - \alpha_2$ in the standard representation. Using the expressions (4.5)-(4.10) in the Hamiltonian $\mathcal{H}_{\mathcal{PT}\text{CMS}}$ in (2.3), the \mathcal{PT} -symmetrically deformed G_2 -CMS potential becomes

$$\begin{aligned} V_{\mathcal{PT}\text{CMS}}^{G_2} = & g_s \sum_{\substack{1 \leq j < k \leq 3 \\ j, k \neq l}} V[R(\varepsilon)(q_j - q_k) + i/3(-1)^{j+k}I(\varepsilon)(q_j + q_k - 2q_l)] \\ & + g_l \sum_{\substack{1 \leq j < k \leq 3 \\ j, k \neq l}} V[(-1)^{j+k+1}R(\varepsilon)(q_j + q_k - 2q_l) + iI(\varepsilon)(q_j - q_k)]. \end{aligned} \quad (4.11)$$

As a result of the aforementioned relation between the A_2 and G_2 -roots the corresponding potentials reduce as $V_{\mathcal{PT}\text{CMS}}^{G_2} \rightarrow V_{\mathcal{PT}\text{CMS}}^{A_2}$, when we switch off the three particle interaction $g_l \rightarrow 0$ and scale the deformation function. When specifying further $R(\varepsilon) = \cosh \varepsilon$ and $I(\varepsilon) = \sqrt{3} \sinh \varepsilon$ we obtain

$$V_{\mathcal{PT}\text{CMS}}^{G_2} = \sum_{k=-1,0,1} g_s V \left[\sqrt{2}r \sin(\phi - i\varepsilon + k\frac{2\pi}{3}) \right] + g_l V \left[\sqrt{6}r \cos(\phi - i\varepsilon + k\frac{2\pi}{3}) \right]. \quad (4.12)$$

Once again we may also choose a different type of deformations according to (3.15), in which the \mathcal{PT} -symmetrically extended model can be brought into the form

$$\begin{aligned} V_{\mathcal{PT}\text{CMS}}^{G_2} = & \frac{g_s}{2} \sum_{1 \leq j < k \leq 3} [V[(R(\varepsilon) + iI(\varepsilon))(q_j - q_k)] + V[(R(\varepsilon) - iI(\varepsilon))(q_j - q_k)]] \\ & + \frac{g_l}{2} \sum_{\substack{1 \leq j < k \leq 3 \\ j, k \neq l}} [V[(R(\varepsilon) + iI(\varepsilon))(q_j + q_k - 2q_l)] + V[(R(\varepsilon) - iI(\varepsilon))(q_j + q_k - 2q_l)]] \end{aligned} \quad (4.13)$$

when choosing the roots to be in the standard representation. We may also express this in terms of Jacobian relative coordinates with the choice $R(\varepsilon) = 1$ and $I(\varepsilon) = \varepsilon/r$ as in the

A_2 -case, such that the potential becomes

$$V_{PT}^{G_2} = \sum_{\substack{k=-1,0,1 \\ n=-1,1}} \frac{g_s}{2} V \left[\sqrt{2}(r + i\varepsilon n) \sin\left(\phi + k\frac{2\pi}{3}\right) \right] + \frac{g_l}{2} V \left[\sqrt{6}(r + i\varepsilon n) \cos\left(\phi + k\frac{2\pi}{3}\right) \right]. \quad (4.14)$$

Let us now study some physical properties of these models.

4.3 Eigensystems

Let us now specialize the potential to the one of the Calogero model, i.e. we take it to be $V(x) \sim 1/x^2$, and determine the eigensystems of the deformed models. In general this is a difficult task as even for the undeformed CMS-models the eigenfunctions are combinations of Vandermonde determinants and Jack polynomials, e.g. [31]. However, in the cases under consideration we can follow a different route and be very explicit for some very particular choices of the deformation functions. As illustrated in the last subsection we may just consider the G_2 -Calogero model and treat the A_2 -Calogero model as a special case of the former by switching off the three particle interaction. The A_2 -model was already solved by Calogero [16] almost forty years ago and the G_2 -case thereafter by Wolfes [30]. Relying on these solutions, the construction of eigensystems for some specific deformed system is fairly simple, as they may be obtained by implementing a shift as was done in the A_2 -case [19]. For other choices of the functions $R(\varepsilon)$ and $I(\varepsilon)$ the solutions can not be constructed in direct analogy to the undeformed case.

However, as was observed in [32, 19] even the simpler scenario is instructive as there are a few differences in the argumentation leading to various constraints on the parameters resulting from the implementation of physical requirements. The main consequence of the deformation is that some irregular solutions, which had to be discarded in the undeformed case become perfectly viable regularized solutions after the deformation. As a result the energy spectra of the deformed systems differ from those of the undeformed ones. Let us briefly recall the argumentation of [16, 30] and treat thereafter the deformed case.

4.3.1 The undeformed case

The above mentioned variable transformations $(x_1, x_2, x_3) \rightarrow (R, X, Y) \rightarrow (R, r, \phi)$ have the virtue that they convert the differential equation into a form allowing for completely separability [16, 30]. The Laplace operator transforms simply as

$$\Delta_{x_1 x_2 x_3} \rightarrow \frac{1}{3} \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \rightarrow \frac{1}{3} \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (4.15)$$

the confining potential transforms as

$$\frac{m^2}{16} \sum_{\alpha \in \Delta} (\alpha \cdot q)^2 \rightarrow \frac{3}{8} m^2 (X^2 + Y^2) \rightarrow \frac{\omega^2}{2} r^2 \quad (4.16)$$

and the Calogero potential as

$$\frac{g_s}{2} \sum_{\alpha \in \Delta_s} \frac{1}{(\alpha \cdot q)^2} \rightarrow \frac{9}{2} g \frac{(X^2 + Y^2)^2}{(X^3 - 3XY^2)^2} \rightarrow \frac{9}{2} \frac{g}{r^2 \sin(3\phi)}, \quad (4.17)$$

$$\frac{g_l}{2} \sum_{\alpha \in \Delta_s} \frac{1}{(\alpha \cdot q)^2} \rightarrow \frac{9}{2} g \frac{(X^2 + Y^2)^2}{(Y^3 - 3YX^2)^2} \rightarrow \frac{9}{2} \frac{g}{r^2 \cos(3\phi)}. \quad (4.18)$$

Assembling these expressions into a Hamiltonian it is then easy to see that in the (R, r, ϕ) -system the eigenfunctions can be factorized into $\Psi(R, r, \phi) = \Phi(R)\chi(r)f(\phi)$, which leads, after separating off the center of mass motion, to the two separate eigenvalue equations

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\lambda^2}{r^2} \right) \chi(r) = E \chi(r), \quad (4.19)$$

$$\left(-\frac{\partial^2}{\partial \phi^2} + \frac{9g_s}{\sin(3\phi)} + \frac{9g_l}{\cos(3\phi)} \right) f(\phi) = \lambda^2 f(\phi). \quad (4.20)$$

These equations may be solved generically for any real values of the parameters $r, \phi, g_s, g_l, \omega$ including even the eigenvalues E and λ^2 by

$$\chi(r) = r^\lambda \exp(-\omega r^2/2) {}_1F_1 \left[\frac{1}{2}(1 + \lambda) - \frac{E}{4\omega}; 1 + \lambda; \omega r^2 \right], \quad (4.21)$$

$$f(\phi) = \sin^{2\kappa_s}(3\phi) \cos^{2\kappa_l}(3\phi) {}_2F_1 \left[\kappa_s + \kappa_l - \frac{\lambda}{6}, \kappa_s + \kappa_l + \frac{\lambda}{6}; 2\kappa_s + \frac{1}{2}; \sin^2(3\phi) \right]. \quad (4.22)$$

Here we abbreviated the constants $\kappa_{s/l} = \kappa_{s/l}^\pm = (1 \pm \sqrt{1 + 4g_{s/l}})/4$, ${}_1F_1$ denotes the Kummer confluent hypergeometric function and ${}_2F_1$ the Gauss hypergeometric function. Implementing now various different physical requirements leads to the quantization condition for the eigenvalues and several restrictions on the parameters

$$P1 : \quad E = 2|\omega|(2n + \lambda + 1) \quad \text{for } n \in \mathbb{N}_0, \quad (4.23)$$

$$P2 : \quad \lambda > 0, \quad (4.24)$$

$$P3 : \quad \kappa_s \rightarrow \kappa_s^+, \kappa_l \rightarrow \kappa_l^+, \quad (4.25)$$

$$P4 : \quad \lambda = 6(\kappa_s + \kappa_l + \ell) \quad \text{for } \ell \in \mathbb{N}_0. \quad (4.26)$$

We briefly recall and extend the argumentations in order to illustrate how they need to be modified in the deformed scenario.

P1 : The quantization condition *P1* originates from the physical requirement that the wavefunction should vanish for $r \rightarrow \infty$. Using the asymptotic expansion for Kummer's confluent hypergeometric function, see e.g. [33],

$${}_1F_1[\alpha; \gamma; z] \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma} G(1-\alpha; \gamma-\alpha, z) \quad \text{for } \operatorname{Re} z > 0, \quad (4.27)$$

$${}_1F_1[\alpha; \gamma; z] \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} (-z)^{-\alpha} G(\alpha; \alpha-\gamma-1, -z) \quad \text{for } \operatorname{Re} z < 0, \quad (4.28)$$

with $G(\alpha; \gamma, z) = 1 + \alpha/z + \alpha(\alpha+1)\gamma(\gamma+1)/2!/z^2 + \dots$, one observes that for the arguments of the solution $\chi(r)$ in (4.21) the function will usually diverge exponentially, unless this divergence is compensated by a diverging gamma function, either from the corresponding $\Gamma(\alpha)$ in (4.27) or $\Gamma(\gamma - \alpha)$ in (4.28). As this is the case when the first argument in ${}_1F_1$ becomes a negative integer, i.e. when the hypergeometric series terminates, the wavefunction $\chi(r)$ vanishes at infinity with the condition $P1$. For these values the Kummer confluent hypergeometric function reduces to a generalized Laguerre polynomial $L_n^\alpha(z)$ by means of the identity

$${}_1F_1[-n; \alpha + 1; z] = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(z) \quad \text{for } n \in \mathbb{N}_0, \alpha \in \mathbb{R} \quad (4.29)$$

and one obtains, up to normalization, the expression for $\chi(r)$ already found by Calogero [16]. Note that this argumentation does not change even if we continue r into the complex plane and $P1$ remains also valid in that case.

P2 : The constraint $P2$ arises from the condition that a proper physical wavefunction should be finite on its domain. In the undeformed case the divergence of $\chi(r)$ at $r = 0$ can be cured by the constraint $P2$. Clearly this constraint can be removed if r acquires a nonvanishing imaginary part, since the factor r^λ no longer diverges for $r \rightarrow 0$.

P3 : The constraint $P3$ results from same requirement as $P2$, but demanding finiteness in the entire domain also for its derivative. For $\kappa_s = \kappa_s^-$ and $\kappa_l = \kappa_l^-$ the prefactors in (4.22) would diverge for $\phi = 0, \pi/3, \dots$ and $\phi = \pi/6, \pi/2, \dots$, respectively. Clearly when $\text{Im } \phi \neq 0$ there is no longer any justification for this constraint and it can be removed, thus allowing the values $\lambda < 0$.

P4 : The quantization condition $P4$ stems from the divergence of $f(\phi)$ at for instance $\phi = \pi/6$. This is seen from the fact that for generic arguments the function ${}_2F_1[\alpha, \beta; \gamma; 1]$ is absolutely convergent when $\text{Re } \gamma > \text{Re}(\alpha + \beta)$, which for the values in (4.22) translates into $\kappa_l < 1/4$. Having already excluded κ_l^- by condition $P3$ this inequality can never be satisfied. However, when α becomes a negative integer the hypergeometric series terminates and reduces to a Jacobi polynomial $P_\ell^{\alpha, \beta}(z)$ by means of the identity

$${}_2F_1[-\ell, \alpha + \beta + \ell + 1; \alpha + 1; z] = \frac{\Gamma(\ell+1)\Gamma(\alpha+1)}{\Gamma(\ell+\alpha+1)} P_\ell^{\alpha, \beta}(1-2z) \quad \text{for } \ell \in \mathbb{N}_0, \alpha, \beta \in \mathbb{R}. \quad (4.30)$$

Since $P_\ell^{\alpha, \beta}(-1) = (\beta+1)_\ell/\ell!$ with $(x)_n := x(x+1)(x+2)\dots(x+n)$ the divergence is removed by condition $P4$. Alternatively we could also equate the second argument in (4.22) to an integer and deduce $\lambda = -6(\kappa_s + \kappa_l + \ell)$, which is however excluded by condition $P2$. Notice that when $\text{Im } \phi \neq 0$, we will even leave the unit circle $|z| \leq 1$, in which convergence can be achieved unless we restrict the real part of ϕ depending on its imaginary part, which seems very artificial. Thus in this case terminating the series by means of property (4.30) appears even more natural than in the undeformed case.

In summary, when the physical constraints $P1, P2, P3, P4$ hold, the corresponding wave functions are

$$\chi_n^\lambda(r) \sim \Gamma(n+1) \omega^{\frac{\lambda}{2}} r^\lambda \exp(-\omega r^2/2) L_n^\lambda(\omega r^2), \quad (4.31)$$

$$f_\ell^{\kappa_s, \kappa_l}(\phi) \sim \Gamma(\ell+1) \sin^{2\kappa_s}(3\phi) \cos^{2\kappa_l}(3\phi) P_\ell^{2\kappa_s-1/2, 2\kappa_l-1/2}[1-2\sin^2(3\phi)]. \quad (4.32)$$

with energy spectrum

$$E_{n\ell} = 2|\omega| [2n + 6(\kappa_s^+ + \kappa_l^+ + \ell) + 1] \quad \text{for } n, \ell \in \mathbb{N}_0. \quad (4.33)$$

Let us now see in a concrete case how the deformation weakens the constraints and how it influences the physics of the models.

4.3.2 The deformed case

We may now consider various types of deformations (3.10) or (3.15) depending in addition on the possible selections for the deformation functions $R(\varepsilon)$ and $I(\varepsilon)$. We consider the deformed G_2 -Calogero model, with the deformation (3.10) and the simplest choice for the deformation functions $R(\varepsilon) = \cosh \varepsilon$ and $I(\varepsilon) = \sqrt{3} \sinh \varepsilon$. This leads to the differential equations (4.19) and (4.20) with a shifted $\phi \rightarrow \phi + i\varepsilon$, i.e. the wavefunctions are simply obtained from (4.31), (4.32) by $\eta_\phi \Psi(R, r, \phi)$ with $\eta_\phi = \exp(p_\phi \varepsilon)$. However, there is a small change in the physical interpretation. From the discussion of the previous subsection follows that $P1, P2$ and $P4$ still have to be implemented on physical grounds, but to demand $P3$ lacks any justification, since the wavefunctions are regularized and no longer diverge. Therefore $P3$ can be relaxed. Consequently we end up with the modified energy spectrum

$$E_{n\ell}^\pm = 2|\omega| [2n + 6(\kappa_s^\pm + \kappa_l^\pm + \ell) + 1] \quad \text{for } n, \ell \in \mathbb{N}_0, \quad (4.34)$$

such that besides the energies $E_{n\ell}^+$ we may now also encounter the energies $E_{n\ell}^-$. Note that we have a degeneracy $E_{n\ell}^+ = E_{n'\ell'}^-$ whenever

$$n' - n + 3(\ell' - \ell) = \frac{3}{2}\sqrt{1+4g_s} + \frac{3}{2}\sqrt{1+4g_l}. \quad (4.35)$$

A similar observation was made for A_2 -Calogero model in [19].

Alternatively we may investigate the deformed G_2 -Calogero model (4.14) based on the deformed roots (3.15) with deformation $R(\varepsilon) = 1$ and $I(\varepsilon) = \varepsilon/r$. The wavefunctions are easy to construct in this case when we break the invariance under the extended Coxeter group $\mathcal{W}^{\mathcal{PT}}$ by restricting the sum in the potential to the positive or negative roots only and scaling the coupling constants g_s, g_l by a factor of 2. Then the corresponding wavefunctions result from (4.31), (4.32) as $\eta_r^\pm \Psi(R, r, \phi)$ with $\eta_r^\pm = \exp(\pm p_r \varepsilon)$. For each of the models the constraints $P1, P3$ and $P4$ still hold on physical grounds, but as the divergence at $r = 0$ for $\chi(r)$ has vanished we no longer have to demand $P2$. This means for both models, that is either extending the roots just over the positive or just over the negative roots, we have the identical energy spectra

$$E_n^\pm = 2|\omega| (2n \pm \lambda + 1), \quad (4.36)$$

thus allowing in addition to E_n^+ also E_n^- . We encounter the degeneracy $E_n^+ = E_{n'}^-$ when $\lambda = n' - n$. Due to the identity

$$z^{m-n}\Gamma(n+1)L_n^{m-n}(z^2) = (-z)^{n-m}\Gamma(m+1)L_m^{n-m}(z^2) \quad (4.37)$$

we find in that situation the wavefunction are related as

$$\chi_n^\lambda(r + i\varepsilon) = (-1)^{n'-n}\chi_{n'}^{-\lambda}(r + i\varepsilon). \quad (4.38)$$

In general, we have the symmetry $\chi_n^\lambda(r) = (-1)^\lambda\chi_n^\lambda(-r)$, such that we can related the wavefunctions of the positive root model $\chi_{n,\text{pos}}^\lambda(r)$ and the negative root model $\chi_{n,\text{neg}}^\lambda(r)$ by an anyonic statistic as $\chi_{n,\text{pos}}^\lambda(r) = (-1)^\lambda\chi_{n,\text{neg}}^\lambda(r)$.

5. Conclusions

We have demonstrated that the Coxeter group represented in \mathbb{R}^n can be deformed in a systematic way to the \mathcal{PT} -symmetrically extended Coxeter group $\mathcal{W}^{\mathcal{PT}}$ represented in $\mathbb{R}^n \oplus i\mathbb{R}^n$. As we have shown there are various ways to achieve this. We may deform the roots across the hyperplanes through the origin orthogonal to each root either by taking the imaginary part to be a vector in this hyperplane (3.10) or a vector orthogonal to it (3.16). As a natural application one may seek for models for which this group constitutes a symmetry group. CMS-models are well known to be invariant under the action of \mathcal{W} and we have demonstrated how they may be deformed such that they remain invariant under the action of $\mathcal{W}^{\mathcal{PT}}$. In fact one simply needs to replace the roots α_i by their deformed counterparts $\tilde{\alpha}_i$. We have worked out the A_2 and G_2 -cases in some detail by constructing explicitly the deformed root systems and applying them thereafter to the corresponding CMS-models. When specializing the deformation functions $R(\varepsilon)$ and $I(\varepsilon)$ in a certain way some easy cases resemble the undeformed case with some simple shifts when transformed to Jacobian relative coordinates, which allowed to determine their corresponding eigensystems. We discussed that as a consequence of the deformation the physical reason leading to some constraints vanishes, such that various restrictions on the parameter space of the model may be relaxed.

Various open challenges remain, as for instance to establish whether the deformations studied here preserve integrability, analogously to what has been established in [20] for the different types of deformation, to investigate models for different choices for the functions $R(\varepsilon)$ and $I(\varepsilon)$ and to study in detail Coxeter groups of higher rank, together with their applications, such as the CMS-models [34]. Models with different choices for the deformation functions will certainly also lead to non-Hermitian Hamiltonians with real spectra, which may be explained by the built in \mathcal{PT} -symmetry [6, 35] or pseudo-Hermiticity [5].

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